Linear differential systems

Rapisarda, Paolo

Chapter 7

Polynomial J-spectral factorization

The problem of polynomial J-spectral factorization of a para-Hermitian matrix arises in different areas of systems and control theory and in signal processing. Its solution is of central importance for example in LQ control theory, in the polynomial and in the behavioral approaches to H_∞-control, and in Wiener filtering.

In this chapter we illustrate an algorithm to perform polynomial J-spectral factorization. The algorithm works under mild assumptions on the matrix to be factored; these assumptions are satisfied by the para-Hermitian matrices arising in polynomial and behavioral H_∞-control problems. The algorithm is based on lifting the problem from the one-variable polynomial context in which it is originally formulated, to a two-variable polynomial context, and on a result of central importance in the approach of Trentelman and Willems to behavioral H_∞-control.

The chapter is organized as follows. We first state the polynomial J-spectral factorization problem and we discuss its connections with H_∞-control in a behavioral setting. We proceed to prove some results of independent interest regarding the connection between Pick matrices and duality of quadratic differential forms. Then we state the main result of this chapter, an algorithm to perform J-spectral factorization. We conclude the chapter with the application of the algorithm to two H_∞-control problems.

The results illustrated in this chapter have been worked out in collaboration with Dr. H. L. Trentelman, and constitute the second part of the paper [42].

7.1 Problem statement

A real polynomial matrix Z is called *para-Hermitian* if $Z^{\sim} = Z$. The problem of *polynomial J-spectral factorization* of a para-Hermitian polynomial matrix is the following. A $q \times q$ para-Hermitian polynomial matrix Z is given, together with two non-negative integers q_+ and q_- such that $q_+ + q_- = q$. A square polynomial matrix F is sought, such that

$$Z = F^{\sim} J_{q_+,q_-} F, \tag{7.1}$$

where J_{q_+,q_-} denotes the signature matrix

$$J_{q_+,q_-} := \begin{pmatrix} I_{q_+} & 0 \\ 0 & -I_{q_-} \end{pmatrix}$$

and where F enjoys certain additional properties besides (7.1). For example, F is most often required to be Hurwitz, meaning that all roots of its determinant are in the open left-half plane, or anti-Hurwitz, meaning that all roots of its determinant are in the open right-half plane. Such an F is called a *Hurwitz*, respectively *anti-Hurwitz spectral factor of Z*. The requirement that the factor F is Hurwitz is typical in spectral factorization problems arising in behavioral LQ control [57] and in the computation of the extremal storage functions for a quadratic differential form examined in Chapter 5 of this thesis. In other cases, and most notably in H_∞-control applications, the para-Hermitian matrix Z is pre-factored as $Z = M^{\sim} J' M$, where J' is some signature matrix; in this setting, the spectral factor F is required to be Hurwitz and such that a number of additional properties are satisfied, for example that MF^{-1} is a matrix of proper rational functions (see [25, 45]).

In order to establish necessary and sufficient conditions for the existence of a Hurwitz J-spectral factorization we must introduce the notion of *signature* of an Hermitian matrix H. This is the ordered triple

$$\text{sign}\,(H) = (n_-(H), n_0(H), n_+(H))$$

where $n_-(H)$ denotes the number (counting multiplicities) of negative eigenvalues of H, $n_0(H)$ denotes the multiplicity of 0 as an eigenvalue of H, and $n_+(H)$ denotes the number (counting multiplicities) of positive eigenvalues of H.

The following result shows that in order to have a Hurwitz J-spectral factorization, the para-Hermitian matrix Z is therefore required to satisfy a *constant signature condition* on the imaginary axis.

Theorem 7.1.1 *Let $Z \in \mathbb{R}^{q \times q}[\xi]$ be a para-Hermitian polynomial matrix. Let q_- and q_+ be two nonnegative integers such that $q_- + q_+ = q$. Assume that $\det(Z(i\omega)) \neq 0$ for all $\omega \in \mathbb{R}$. The following statements are equivalent:*

(i) sign $(Z(i\omega)) = (q_-, 0, q_+)$ for all $\omega \in \mathbb{R}$

(ii) There exists a $q \times q$ Hurwitz matrix F such that

$$Z = F^{\sim} J_{q_+,q_-} F$$

Proof: See [33], Section 5. □

Note that in the case $q_- = 0$, i.e. the classical spectral factorization problem arising in LQ control theory and in filtering, the signature condition corresponds to positive definiteness of Z on the imaginary axis.

Due to the central importance of the problem in many areas of application, several algorithms have been developed to perform J-spectral factorization. They are based on various techniques, among the others successive factor extraction (see [8, 24]), the Newton-Raphson method (see [19]), and the algebraic Riccati equation approach (see [1, 42]). For a review of existing algorithms we refer to [24].

In the next section we illustrate a necessary and sufficient condition for a para-Hermitian matrix of the form $M^{\sim} J' M$ to have a Hurwitz J-spectral factor F that satisfies certain additional requirements. In order to state this result and to motivate the additional requirements on the spectral factor, we will first illustrate the background of this version of the J-spectral factorization problem.

7.2 H_∞-control in a behavioral setting

The basic problem of classical control theory is that of designing a feedback controller such that the closed loop system meets certain design objectives. The system to be controlled has variables on which the controller can act (the control inputs) and variables which are to be controlled (the measured outputs). The system is also subject to disturbances, both in the input and in the output variables.

While successful in a large area of application, the classical point of view fails to accommodate control problems in which the distinction between input and output variables is blurred, or altogether absent; and in which the system is not subject to disturbances in the control inputs and the measurement outputs, because no such classification can be imposed on the variables describing the system. Examples of such situations are provided in [23, 56]. In this section we illustrate a different approach to control, based on behavioral ideas. We will center the exposition on the basic ideas of full information H_∞-control in a behavioral setting, referring to [23, 56, 45] for a thorough exposition.

Behavioral control theory centers around the notion of the plant as a set of trajectories which the control system constrains in order to meet the control objectives. Controller design is viewed as the problem of devising and implementing a set of equations that the trajectories of the plant must satisfy, in such a way that the controlled system, associated with the subset of the behavior of the plant consisting of the trajectories that also satisfy the equations imposed by the controller, meets the design objectives. In most cases the plant and the

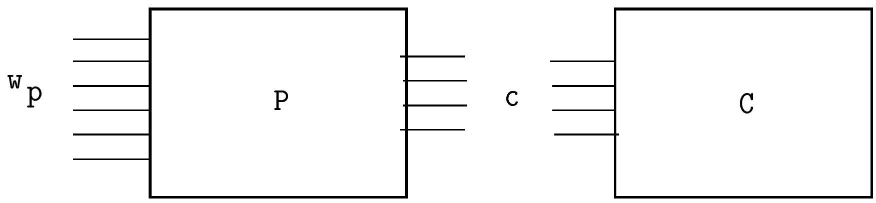

Figure 7.1: The plant and the controller

controller do not interact directly through their manifest variables, and the systems are interconnected through certain terminals. We call the variables associated with these terminals the *interconnection variables*. By influencing the interconnection variables, the controller then affects the whole behavior of the plant.

We now formalize this point of view on control in the language of behaviors. The plant is a dynamical system $\Sigma_p = (T, W_p \times C, \mathfrak{B}_p)$, whose signal space is partitioned in the external variable space W_p and the interconnection variable space C. The controller is a system $\Sigma_c = (T, C, \mathfrak{B}_c)$ which has the same time set and the same interconnection variable set of the plant (see Figure 7.2). The *interconnection of the plant and the controller* (depicted in Figure 7.2) is the system

$$\Sigma_p \wedge \Sigma_c = (T, W_p, \mathfrak{B})$$

where

$$\mathfrak{B} = \{w_p \text{ s.t. } c \in \mathfrak{B}_c \text{ and } (w_p, c) \in \mathfrak{B}_p\}.$$

The problem is to design Σ_c in such a way that the trajectories in $\mathfrak{B}$ enjoy certain desirable properties, expressed by control objectives. In or-

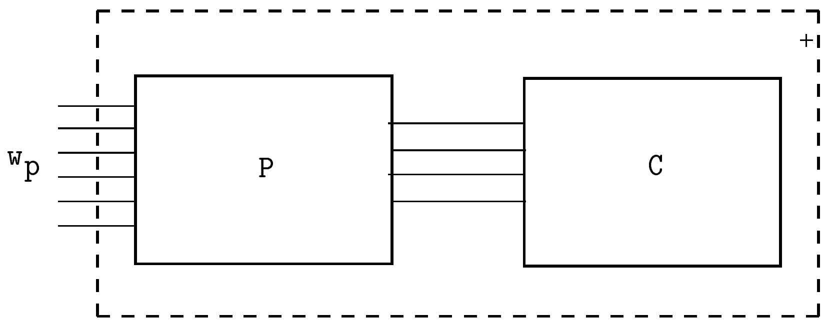

Figure 7.2: The interconnection of plant and controller

der to specify the control objectives, two different approaches have been developed in behavioral system theory. The first one is due to Stoorvogel and Weiland, and consists in requiring that the controlled system behavior satisfies a minimal and a maximal requirement, specified by means of sub- and supersets of the controlled system behavior. In this setting, a control objective is a quadruple $(\mathcal{R}_{min}, \mathcal{S}_{min}, \mathcal{R}_{max}, \mathcal{S}_{max})$ of subsets of $(\mathbb{R}^q)^{\mathbb{R}}$. A controller is said to achieve the control objective if the behavior $\mathfrak{B}$ of $\Sigma_p \wedge \Sigma_c$ satisfies

$$\begin{aligned} \mathcal{S}_{min} &\subseteq \mathfrak{B} + \mathcal{R}_{min} \\ \mathfrak{B} \cap \mathcal{R}_{max} &\subseteq \mathcal{S}_{max} \end{aligned}$$

Consequently, the controlled system behavior must be "rich enough", meaning that a suitable extension of $\mathfrak{B}$ contains a specified set of trajectories, and it must satisfy a performance requirement, expressed by asking that a suitable subset of $\mathfrak{B}$ exhibits a certain desired behavior. This point of view on behavioral control theory has been applied to ℓ_1, H_2 and H_∞-control problems; we refer the interested reader to [39, 49] for a thorough exposition.

The second approach to the specification of the control objectives is due to Trentelman and Willems and relies on the calculus of QDF's. The results on polynomial J-spectral factorization illustrated in the next section stem from this approach. Consequently, we will introduce it in detail by describing the formalization of the full information H_∞-control problem.

In the full information H_∞-control problem, the desired property of the controlled system is that certain components of the external variables, called the *to-be-controlled variables*, denoted z, are "small" regardless of the values taken by other components of the external variables, called the *disturbances*, denoted d. The disturbances are unknown and any disturbance trajectory is possible, i.e. the variable d is free. The requirement that the to-be-controlled variable be small with respect to the disturbance variable is expressed by requiring that for all square integrable trajectories (z, d) in the controlled system there holds $\|z\|_2 \leq \gamma \|d\|_2$, where $\|\cdot\|_2$ is the $\mathfrak{L}_2$-norm and γ is a desired performance level. It is also required that the controlled system is *stable*, meaning that in absence of disturbances the signal z goes to zero as t tends to infinity. The control objective must be attained acting on certain interconnection variables c of the plant. The plant is therefore a system $\Sigma_p = (\mathbb{R}, \mathbb{R}^z \times \mathbb{R}^d \times \mathbb{R}^c, \mathfrak{B}_p)$, where z, d, and c denote also the dimension of the corresponding variables. The system is assumed to be controllable and therefore admits an observable image representation

$$\begin{pmatrix} z \\ d \\ c \end{pmatrix} = \begin{pmatrix} Z(\frac{\mathrm{d}}{\mathrm{d}t}) \\ D(\frac{\mathrm{d}}{\mathrm{d}t}) \\ C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{pmatrix} \ell \tag{7.2}$$

where ℓ is a l-dimensional variable. In this setting, z, d, c, and ℓ are assumed to be infinitely differentiable functions. Note that since d is a free variable, D has full row rank.

A controller is a system $\Sigma_c = (\mathbb{R}, \mathbb{R}^c, \mathfrak{B}_c)$ represented as

$$K(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$$

or in an equivalent way (see [45]) as

$$c = C(\frac{\mathrm{d}}{\mathrm{d}t})\ell \tag{7.3}$$

$$0 = K'(\frac{\mathrm{d}}{\mathrm{d}t})\ell. \tag{7.4}$$

When it is interconnected with the plant, the controller gives rise to a controlled system $\Sigma_p \wedge \Sigma_c = (\mathbb{R}, \mathbb{R}^z \times \mathbb{R}^d, \mathfrak{B}_p \wedge \mathfrak{B}_c)$ where

$$\mathfrak{B}_p \wedge \mathfrak{B}_c = \{(z,d) \text{ s.t. } c \in \mathfrak{B}_c \text{ and } (z,d,c) \in \mathfrak{B}_p\}$$

A controller is called an *admissible γ-contracting stabilizing controller* if the following three properties hold for $\Sigma_p \wedge \Sigma_c$:

(1) d is a free variable, i.e. for all $d \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^d)$ there exists $z \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^z)$ such that $(z,d) \in \mathfrak{B}_p \wedge \mathfrak{B}_c$

(2) $\{(z,0) \in \mathfrak{B}_p \wedge \mathfrak{B}_c\} \Longrightarrow \{\lim_{t\to\infty} z(t) = 0\}$

(3) $\forall\ (z,d) \in (\mathfrak{B}_p \wedge \mathfrak{B}_c) \cap \mathfrak{L}_2(\mathbb{R}, \mathbb{R}^{z+d})$ there holds

$$||z||_2 \leq \gamma ||d||_2$$

The first property requires that any disturbance can occur in the controlled system, i.e. the variable d remains free also after interconnection; the second one requires stability of the controlled system in absence of disturbances, while the third one specifies an upper bound on the H_∞-performance required of the controlled system. A controller is called an *admissible strictly γ-contracting stabilizing controller* if properties (1) – (2) above hold, and moreover there exists $\epsilon > 0$ such that property (3) above holds with γ replaced by $\gamma - \epsilon$.

In order to state a necessary and sufficient condition on the existence of an admissible strictly γ-contracting stabilizing controller, we introduce the polynomial matrix

$$M(\xi) = \begin{pmatrix} Z(\xi) \\ D(\xi) \end{pmatrix}$$

which, together with the diagonal matrix

$$\Sigma_\gamma := \begin{pmatrix} I_z & 0 \\ 0 & -\gamma^2 I_d \end{pmatrix}$$

defines the symmetric two-variable polynomial matrix $\Phi_\gamma(\zeta, \eta)$

$$\Phi_\gamma(\zeta, \eta) := M^T(\zeta)\Sigma_\gamma M(\eta).$$

In the following we assume that M is observable. We denote by T_{Φ,S_-} the Pick matrix of $\Phi_\gamma(\zeta,\eta)$ associated with the open left half-plane roots of $\det(\partial\Phi_\gamma)$.

The following result is a necessary and sufficient condition for the existence of an admissible strictly γ-contracting stabilizing controller.

Theorem 7.2.1 *Let $\gamma > 0$. The following statements are equivalent:*

1. *there exists a stabilizing, strictly γ-contracting controller for (7.2)*
2. *The following two conditions hold*
   (a) *there exists $\epsilon > 0$ such that for all $\omega \in \mathbb{R}$ there holds*

   $$\text{sign}\ (\partial\Phi_\gamma(i\omega) + \epsilon M^T(-i\omega)M(i\omega)) = (d, 0, l-d)$$

   (b) $T_{\Phi,S_-} < 0$
3. *there exists $F \in \mathbb{R}^{l\times l}[\xi]$ such that*
   (a) $\partial\Phi_\gamma = F^\sim J_{l-d,d} F$

(b) F *is Hurwitz*

(c) MF^{-1} *is a matrix of proper rational functions*

(d) $\begin{pmatrix} D \\ F_+ \end{pmatrix}$ *is Hurwitz*

Here F_+ *is obtained by partitioning* F *into*

$$\begin{pmatrix} F_+ \\ F_- \end{pmatrix}$$

where F_+ *has* $l-d$ *rows and* F_- *has* d *rows. If* F *satisfies* (3) *above, then the controller specified as*

$$\begin{aligned} c &= C(\frac{\mathrm{d}}{\mathrm{d}t})\ell \\ 0 &= F_+(\frac{\mathrm{d}}{\mathrm{d}t})\ell \end{aligned}$$

is admissible, strictly γ*-contracting and stabilizing.*

Proof: See Theorem 11.1 of [45]. □

We now go back to the J-spectral factorization problem. We first give the following definition.

Definition 7.2.1 *Let the para-Hermitian matrix* $Z \in \mathbb{R}^{q\times q}[\xi]$ *be given by* $Z = M^{\sim}J_{p_+,p_-}M$*, where* $M \in \mathbb{R}^{(p_++p_-)\times q}[\xi]$*,* $M(\lambda)$ *is of full column rank for all* $\lambda \in \mathbb{C}$*, and* $p_+ + p_- > q$*.* $F \in \mathbb{R}^{q\times q}[\xi]$ *is called a* regular, Hurwitz, stabilizing J_{q-p_-,p_-}-spectral factor *of* Z *if:*

1. $Z = F^{\sim}J_{q-p_-,p_-}F$

2. F *is Hurwitz*

3. MF^{-1} *is proper*

4. *The matrix* $\begin{pmatrix} M_- \\ F_+ \end{pmatrix}$ *is Hurwitz. Here* M_- *consists of the last* p_- *rows of* M *and* F_+ *consists of the first* $q-p_-$ *rows of* F*.*

The following result is an immediate consequence of Theorem 7.2.1.

Theorem 7.2.2 *Let the para-Hermitian matrix $Z \in \mathbb{R}^{q\times q}[\xi]$ be factored as $Z = M^{\sim} J_{p_+,p_-} M$, where $M \in \mathbb{R}^{(p_+ + p_-)\times q}[\xi]$, $M(\lambda)$ is of full column rank for all $\lambda \in \mathbb{C}$, and $p_+ + p_- > q$. Define $\Phi(\zeta,\eta) = M^T(\zeta) J_{p_+,p_-} M(\eta)$. The following statements are equivalent:*

1. *Z admits a regular Hurwitz stabilizing J_{q-p_-,p_-}-spectral factorization;*

2. *The following conditions hold:*

   (a) *there exists $\epsilon > 0$ such that for all $\omega \in \mathbb{R}$ there holds*

   $$\textit{sign}\ (Z(-i\omega) + \epsilon M^T(-i\omega) M(i\omega)) = (p_-, 0, q - p_-)$$

   (b) *The Pick matrix T_{Φ,S_-} of $\Phi(\zeta,\eta) := M^T(\zeta) J_{p_+,p_-} M(\eta)$ is negative definite.*

Proof: Take $\gamma = 1$, $z = p_+$, $d = p_-$, $l = q$ in Theorem 7.2.1, and observe that statement (1) of the present theorem is equivalent to statement (3) in Theorem 7.2.1. $\square$

In the following we will refer to the condition $(2.a)$ above as the *strict signature condition.*

The rest of this chapter is devoted to the illustration of a technique to compute a regular Hurwitz stabilizing J-spectral factorization for a para-Hermitian matrix $Z = M^{\sim} J' M$ satisfying $(2.a)$ and $(2.b)$ of Theorem 7.2.2.

7.3 An algorithm for J-spectral factorization

In this section we state the main result of the chapter, a formula that allows the computation of a regular, Hurwitz, stabilizing J-spectral factorizations of a matrix satisfying properties $(2.a)$ and $(2.b)$ in Theorem 7.2.2. In order to state the result, we need several intermediate results regarding duality of QDF's and its connections with Pick matrices, that are interesting in their own right.

We begin by proving that $\det(\partial\Phi)$ and $\det(\partial\Phi^\perp)$ have the same roots.

Lemma 7.3.1 *Let p_- and p_+ be nonnegative integers, and let q be a positive integer such that $p_- + p_+ > q$. Let $M \in \mathbb{R}^{(p_- + p_+)\times q}[\xi]$ be such that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, and let $R \in \mathbb{R}^{(p_+ + p_- - q)\times(p_- + p_+)}[\xi]$ be such that $RM = 0$ and $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Let $\Phi(\zeta,\eta) := M^T(\zeta)J_{p_+,p_-}M(\eta)$ and $\Phi^\perp(\zeta,\eta) := R(-\zeta)J_{p_+,p_-}R^T(-\eta)$. Assume that $\partial\Phi$ is semi-simple.*

Then for all $\lambda \in \mathbb{C}$ we have $\det(\partial\Phi(\lambda)) = 0$ *iff* $\det(\partial\Phi^\perp(\lambda)) = 0$. *For any such λ we have* $\dim(\mathrm{Ker}\ \partial\Phi(\lambda)) = \dim(\mathrm{Ker}\ \partial\Phi^\perp(\lambda))$.

Proof: See appendix. □

We now recall the following result.

Lemma 7.3.2 *Let R, M, Φ, and $\Phi^\perp$ be as in Lemma 7.3.1. The following statements are equivalent:*

1. *There exists $\epsilon > 0$ such that for all $\omega \in \mathbb{R}$*

$$\textit{sign}\ (\partial\Phi(i\omega) + \epsilon M^T(-i\omega)M(i\omega)) = (p_-, 0, q - p_-)$$

2. *There exists $\epsilon' > 0$ such that for all $\omega \in \mathbb{R}$*

$$\partial\Phi^\perp(i\omega) \geq \epsilon' R(i\omega)R^T(-i\omega).$$

Proof: See the equivalence (1) $\Longleftrightarrow$ (2) in Theorem 10.1 of [45]. □

Consequently, the strict signature condition on Φ is equivalent to the strict positivity of $\Phi^\perp$.

The following result states that the strict signature condition on $\partial\Phi$ implies that if $\partial\Phi$ is semisimple, then $\partial\Phi^\perp$ is semisimple.

Lemma 7.3.3 *Let R, M, Φ, and $\Phi^\perp$ be as in Lemma 7.3.1. Assume that there exists $\epsilon > 0$ such that for all $\omega \in \mathbb{R}$*

$$\textit{sign}\ (\partial\Phi(i\omega) + \epsilon M^T(-i\omega)M(i\omega)) = (p_-, 0, q - p_-)$$

Then $\deg(\det\partial\Phi) = \deg(\det\partial\Phi^\perp)$. *Moreover, $\partial\Phi$ is semi-simple iff $\partial\Phi^\perp$ is semi-simple.*

Proof: See appendix. □

We are now in the position of stating the first relevant result of this section, namely that if a QDF Q_Φ is strictly average positive and if $\Phi^\perp$ is its dual, then their Pick matrices associated with the left half-plane roots of $\det(\partial\Phi)$ and $\det(\partial\Phi^\perp)$ coincide.

Theorem 7.3.1 *Let p_- and p_+ be nonnegative integers, and let q be a positive integer such that $p_- + p_+ > q$. Let $M \in \mathbb{R}^{(p_-+p_+)\times q}[\xi]$ be such that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, and let $R \in \mathbb{R}^{(p_++p_--q)\times(p_-+p_+)}[\xi]$ be such that $RM = 0$ and $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Define $\Phi(\zeta,\eta) := M^T(\zeta)J_{p_+,p_-}M(\eta)$ and $\Phi^\perp(\zeta,\eta) := R(-\zeta)J_{p_+,p_-}R^T(-\eta)$. Assume that there exists $\epsilon > 0$ such that for all $\omega \in \mathbb{R}$ there holds*

$$sign(\partial\Phi(i\omega) + \epsilon M^T(-i\omega)M(i\omega)) = (p_-, 0, q - p_-)$$

and assume that $\partial\Phi$ is semi-simple. Let $\lambda_1, \lambda_2, \ldots \lambda_k$ be the distinct roots of $\det(\partial\Phi)$ *in $\mathbb{C}^-$; denote* $\dim(\text{Ker }(\partial\Phi(\lambda_i))$ *with n_i, $i \in \underline{k}$ and let $V_i \in \mathbb{C}^{q\times n_i}$ be full column rank matrices such that*

$$\text{Im }(V_i) = \text{Ker }(\partial\Phi(\lambda_i))$$

$i \in \underline{k}$. Then there exist full column rank matrices $V_i' \in \mathbb{C}^{(p_-+p_+-q)\times n_i}$ such that

$$J_{p_+,p_-}R^T(-\lambda_i)V_i' = M(\lambda_i)V_i, \qquad i \in \underline{k}$$

For such matrices V_i' we have $\text{Im }(V_i') = \text{Ker }(\partial\Phi^\perp(\lambda_i))$, *$i \in \underline{k}$. Furthermore, for $i, j \in \underline{k}$ we have*

$$\frac{V_i^*\Phi(\bar\lambda_i,\lambda_j)V_j}{\bar\lambda_i + \lambda_j} = \frac{V_i'^*\Phi^\perp(\bar\lambda_i,\lambda_j)V_j'}{\bar\lambda_i + \lambda_j}$$

i.e., the Pick matrices T_Φ and $T_{\Phi^\perp}$ associated with $\{\lambda_1, \lambda_2, \ldots \lambda_k\}$ are the same.

Proof: See appendix. □

We proceed to derive a useful identity that shows how the Pick matrix of a strictly positive quadratic differential form can be computed in terms of a matched pair of state maps (see section 4.3).

Theorem 7.3.2 *Let p_- and p_+ be nonnegative integers, and let q be a positive integer such that $p_- + p_+ > q$. Let $M \in \mathbb{R}^{(p_-+p_+)\times q}[\xi]$ be such that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, and let $R \in \mathbb{R}^{(p_++p_--q)\times(p_-+p_+)}[\xi]$ be such that $RM = 0$ and $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Let $\Phi(\zeta,\eta) := M^T(\zeta)J_{p_+,p_-}M(\eta)$ and $\Phi^\perp(\zeta,\eta) := R(-\zeta)J_{p_+,p_-}R^T(-\eta)$. Assume that there exists $\epsilon > 0$ such that for all $\omega \in \mathbb{R}$ there holds*

$$sign\ (\partial\Phi(i\omega) + \epsilon M^T(-i\omega)M(i\omega)) = (p_-, 0, q - p_-)$$

and assume that $\partial\Phi$ is semisimple. Let $\lambda_1, \dots \lambda_k$ be the distinct roots of $\det(\partial\Phi)$ in $\mathbb{C}_-$, and let $V_i \in \mathbb{C}^{q\times n_i}$ be full column rank matrices such that $\mathrm{Im}\ (V_i) = \mathrm{Ker}\ (\partial\Phi(\lambda_i))$ $i \in \underline{k}$. Let $V_i' \in \mathbb{C}^{(p_-+p_+-q)\times n_i}$ be full column rank matrices such that

$$J_{p_+,p_-}R^T(-\lambda_i)V_i' = M(\lambda_i)V_i$$

$i \in \underline{k}$. Let T_ϕ be the Pick matrix of Φ associated with $\{\lambda_i\}_{i\in\underline{k}}$ and the V_i's. For any matched pair of state maps (X, Z) for M and $R^\sim$ there holds

$$W^*V = T_\Phi \tag{7.5}$$

where V and W are the V-matrices of (Φ, X) and $(\Phi^\perp, Z)$ respectively, i.e. the complex $n \times n$ matrices defined by

$$V := (\ X(\lambda_1)V_1 \quad X(\lambda_2)V_2 \quad \dots \quad X(\lambda_k)V_k\ )$$

and

$$W := (\ Z(\lambda_1)V_1' \quad Z(\lambda_2)V_2' \quad \dots \quad Z(\lambda_k)V_k'\ )$$

Proof: See appendix. □

We can now state the main result of this chapter.

Theorem 7.3.3 *Let the para-Hermitian matrix $Z \in \mathbb{R}^{q\times q}[\xi]$ be given by $Z = M^\sim J_{p_+,p_-}M$, where $M \in \mathbb{R}^{(p_++p_-)\times q}[\xi]$, $M(\lambda)$ is of full column rank for all $\lambda \in \mathbb{C}$, and $p_+ + p_- > q$. Assume that det(Z) has at least one root. Let $\Phi(\zeta,\eta) = M^T(\zeta)J_{p_+,p_-}M(\eta)$. Assume that $\partial\Phi$ is semisimple, and that there exists $\epsilon > 0$ such that for all $\omega \in \mathbb{R}$*

$$sign\ (\partial\Phi(i\omega) + \epsilon M^T(-i\omega)M(i\omega)) = (p_-, 0, q - p_-)$$

Let $\lambda_1, \ldots, \lambda_k$ be the distinct roots of $\det(\partial\Phi)$ *in* $\mathbb{C}_-$*, with multiplicity* n_i*. Then* $n := \sum_{i=1}^{k} n_i$ *equals the McMillan degree of* M*, denoted by* $n(M)$*, and* $\frac{1}{2}\deg(Z)$*. Let* $X \in \mathbb{R}^{n\times q}[\xi]$ *define a minimal state map for* M*. Let* $V_i \in \mathbb{C}^{q\times n_i}$ *be a full column rank matrix such that* $\mathrm{Ker}\ (\partial\Phi(\lambda_i)) = \mathrm{Im}\ (V_i)$*,* $i = 1, \ldots, k$*. Let* T_Φ *be the Pick matrix of* Φ *associated with* $\{\lambda_i\}_{i\in\underline{k}}$ *and the* V_i*'s. Assume that* $\det(T_\Phi) \neq 0$*. Then the* V*-matrix of* (Φ, X)*, i.e.*

$$V := \begin{pmatrix} X(\lambda_1)V_1 & X(\lambda_2)V_2 & \ldots & X(\lambda_k)V_k \end{pmatrix}$$

is nonsingular. Under these conditions there exists $F \in \mathbb{R}^{q\times q}[\xi]$ *such that*

$$\begin{aligned} M^T(\zeta)J_{p_+,p_-}M(\eta) - (\zeta+\eta)X^T(\zeta)(V^*)^{-1}T_\Phi V^{-1}X(\eta) = \\ F^T(\zeta)J_{q-p,p_-}F(\eta) \qquad (7.6) \end{aligned}$$

Moreover, for any $F \in \mathbb{R}^{q\times q}[\xi]$ *such that (7.6) holds we have* $\det(F) \neq 0$ *and* MF^{-1} *is a matrix of proper rational functions. If, in addition,* $T_\Phi < 0$*, then for any* F *such that (7.6) holds,* F *and* $\mathrm{col}\ (M_-, F_+)$ *are Hurwitz. Here* M_- *and* F_+ *are obtained by taking the last* p_- *and the first* $q - p_-$ *rows of* M *and* F*, respectively.*

Proof: See appendix. □

Observe that under the conditions of Theorem 7.3.3, any F satisfying (7.6) is a regular, stabilizing, Hurwitz J-spectral factor. Indeed, by taking $\zeta = -\xi$ and $\eta = \xi$ in equation (7.6) we obtain the J-spectral factorization $M^T(-\xi)J_{p_+,p_-}M(\xi) = F^T(-\xi)J_{q-p_-,p_-}F(\xi)$. We now show how to compute effectively F from (7.6). Recall from Section 4.1 the definition of the downward and of the right shift σ_D and σ_R. In the following we will denote with $\tilde{\Phi}$ the coefficient matrix of $\Phi(\zeta, \eta)$, and with $\tilde{X}$ and $\tilde{F}$ the coefficient matrices of $X(\xi)$ and $F(\xi)$, i.e.

$$\begin{aligned} X(\xi) &= \tilde{X}\ \mathrm{col}\ (I_q, I_q\xi, \ldots) \\ F(\xi) &= \tilde{F}\ \mathrm{col}\ (I_q, I_q\xi, \ldots). \end{aligned}$$

Observe that $\Phi(\zeta, \eta) \in \mathbb{R}_s^{q\times q}[\zeta, \eta]$ is zero if and only if its coefficient matrix $\tilde{\Phi}$ is zero. Consequently, (7.6) is equivalent with

$$\begin{aligned} \tilde{M}^T J_{p_+,p_-}\tilde{M} - \sigma_R(\tilde{X}^T(V^*)^{-1}T_\Phi V^{-1}\tilde{X}) - \\ \sigma_D(\tilde{X}^T(V^*)^{-1}T_\Phi V^{-1}\tilde{X}) = \tilde{F}^T J_{q-p,p_-}\tilde{F} \qquad (7.7) \end{aligned}$$

Therefore, $\tilde{F}$ can be computed from the factorization of the *constant* matrix on the left hand side of equation (7.7). Observe that since the effective size of the matrices on the left hand side of (7.7) is finite, the factorization can be performed by standard numerical methods.

We conclude this section by spelling out an algorithm to compute a regular, Hurwitz, stabilizing J-spectral factor of a para-Hermitian matrix Z given by $Z = M^{\sim} J' M$.

Algorithm 7.3.1

Input: *A semisimple para-Hermitian $q \times q$ matrix Z given by $Z = M^{\sim} J_{p_-,p_+} M$, where it is assumed that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$ and that* $\det(Z(i\omega)) \neq 0$ *for all $\omega \in \mathbb{R}$.*

Output: *A $q \times q$ regular, Hurwitz, stabilizing J_{q-p_-,p_-}-spectral factor F, if it exists*

Step 1: *Check whether there exists $\epsilon > 0$ such that*

$$\textit{sign}\ (Z(i\omega) + \epsilon M^T(-i\omega)M(i\omega)) = (p_-, 0, q - p_-)$$

for all $\omega \in \mathbb{R}$. If yes, go to step 2, otherwise terminate: no suitable F exists.

Step 2: *Compute the distinct roots $\lambda_1, \lambda_2, \ldots, \lambda_k$ of* $\det(Z)$ *in $\mathbb{C}^-$, and compute matrices $V_i \in \mathbb{C}^{q \times n_i}$ of full column rank such that* $\mathrm{Im}\ (V_i) = \mathrm{Ker}\ (Z(\lambda_i))$. *Compute the Pick matrix*

$$T_\Phi = \left(\frac{1}{\bar{\lambda}_i + \lambda_j} V_i^* M^T(\bar{\lambda}_i) J_{p_+,p_-} M(\lambda_j) V_j \right)_{i,j=1,\ldots,k}$$

Check whether $T_\Phi < 0$. If yes, go to step 3, otherwise terminate: no suitable F exists.

Step 3: *Let $n = \sum_{i=1}^{k} n_i$, and compute a $n \times q$ polynomial matrix X such that X is a state map for M.*

Step 4: *Compute the complex $n \times n$ matrix*

$$V = \begin{pmatrix} X(\lambda_1)V_1 & X(\lambda_2)V_2 & \ldots & X(\lambda_k)V_k \end{pmatrix}$$

Step 5: *The hermitian matrix*

$$\tilde{M}^T J_{p_+,p_-} \tilde{M} - \sigma_R(\tilde{X}^T (V^*)^{-1} T_\Phi V^{-1} \tilde{X}) - \sigma_D(\tilde{X}^T (V^*)^{-1} T_\Phi V^{-1} \tilde{X})$$

has p_- negative eigenvalues and $q - p_-$ positive eigenvalues. Factorize it as $\tilde{F}^T J_{q-p_-,p_-} \tilde{F}$.

Step 6: *Let $F(\xi) := \tilde{F}$ col $(I_q, I_q\xi, I_q\xi^2, \ldots)$.*

7.4 Examples

We now illustrate the application of Algorithm 7.3.1 with two examples.

Example 7.4.1 We consider the example of a mixed sensitivity problem from [25], Example 4.4.3, with parameters $r = 0$, $c = 1$, and $\gamma = 2$. Let

$$M(\xi) := \begin{pmatrix} 1 & -1-\xi \\ -1 & 0 \\ 2 & -2\xi \end{pmatrix}$$

and therefore $q = 2$. Take $p_- = 2$ and $p_+ = 1$, and consider

$$Z(\xi) = M^T(-\xi) J_{2,1} M(\xi) = \begin{pmatrix} -2 & -1+3\xi \\ -1-3\xi & 1+3\xi^2 \end{pmatrix}$$

The two-variable polynomial matrix Φ associated with Z is

$$\Phi(\zeta,\eta) = M^T(\zeta) J_{2,1} M(\eta) = \begin{pmatrix} -1 & -1+3\eta \\ -1+3\zeta & 1+\zeta+\eta-3\zeta\eta \end{pmatrix}$$

It can be verified that $\partial\Phi(i\omega) + \epsilon M^T(-i\omega)M(i\omega)$ has constant signature $(1,0,1)$ for $\epsilon > 0$ sufficiently small. Moreover $\det(Z(\xi)) = 3(\xi^2 - 1)$, so the only left half-plane root of $\det(Z)$ is -1, with multiplicity 1. Since the roots of $\det(Z)$ are simple, Z is semisimple. The kernel of $Z(-1)$ is spanned by $(-2 \;\; 1)^T$, and the Pick matrix of Φ associated with $\{-1\}$ is the 1×1 matrix

$$T_\Phi = \frac{1}{-2} \begin{pmatrix} -2 & 1 \end{pmatrix} \begin{pmatrix} -2 & -4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = -2$$

Note that T_Φ is negative definite; we conclude that there exists a regular, Hurwitz, stabilizing $J_{1,1}$-factor F of Z. We now compute it. A minimal state map for M can be chosen to be $X(\xi) = \begin{pmatrix} 0 & 1 \end{pmatrix}$, so

$$V = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 1$$

We now compute $\tilde{F}^T J_{1,1} \tilde{F}$ as

$$\begin{aligned} \tilde{M}^T J_{2,1} \tilde{M} \quad - \quad & \sigma_R(\tilde{X}^T (V^*)^{-1} T_\Phi V^{-1} \tilde{X}) - \\ & \sigma_D(\tilde{X}^T (V^*)^{-1} T_\Phi V^{-1} \tilde{X}) = \begin{pmatrix} -2 & 1 & 0 & 3 \\ -1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & -3 \end{pmatrix} \end{aligned}$$

Indeed, this matrix has $p_1 = 1$ negative eigenvalues and $q - p_- = 1$ positive eigenvalues. Therefore it can be factored as $\tilde{F}^T J_{1,1} \tilde{F}$, with

$$\tilde{F} = \begin{pmatrix} -0.26 & -1.37 & 0 & -0.86 \\ 1.44 & 0.94 & 0 & -1.93 \end{pmatrix}$$

Consequently, a regular, stabilizing, Hurwitz $J_{1,1}$-spectral factor $F(\xi)$ is

$$F(\xi) = \begin{pmatrix} -0.26 & -1.37 - 0.86\xi \\ 1.44 & 0.94 - 1.93\xi \end{pmatrix}.$$

$\square$

Example 7.4.2 We now consider a factorization problem connected with the computation of the optimal suspension of a vehicle that drives on a bumpy road (see [7], chapter 5). The system to be controlled has four external variables, one of which is a disturbance, and two latent variables. The equations of the system are

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ d \end{pmatrix} = \begin{pmatrix} 0 & \frac{\mathrm{d}^3}{\mathrm{d}t^3} \\ 0 & \frac{\mathrm{d}^2}{\mathrm{d}t^2} \\ -1 & 1 \\ 1 + \alpha \frac{\mathrm{d}}{\mathrm{d}t} & 0 \end{pmatrix} \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}$$

Here α is a design parameter. Observe that for all values of $\lambda \in \mathbb{C}$ the matrix $M(\lambda)$ has full column rank.

The para-Hermitian matrix to be factored is

$$Z(\xi) = M^T(-\xi) \begin{pmatrix} I_3 & 0 \\ 0 & -\gamma \end{pmatrix} M(\xi) =$$
$$\begin{pmatrix} 1 + (-1 + \alpha^2\, \xi^2)\, \gamma & -1 \\ -1 & 1 + \xi^4 - \xi^6 \end{pmatrix}$$

It can be verified that for $\alpha = 1$ and $\gamma = 0.7$ the matrix $Z(i\omega)$ has one negative eigenvalue and one positive eigenvalue for all $\omega \in \mathbb{R}$. This property is unaffected by small perturbations of the matrix, and therefore Z enjoys the strict signature condition. Define $\Phi(\zeta, \eta) = M^T(\zeta) J_{3,1} M(\eta)$. The roots of $\det(\partial\Phi)$ are

$$\begin{matrix} -1 & -0.936 & -0.456 - 0.927i & -0.456 + 0.927i \\ 1 & 0.936 & 0.456 + 0.927i & 0.456 - 0.927i \end{matrix}$$

and since they are all distinct, the matrix $\partial\Phi(\xi)$ is semisimple. In the following we will denote the roots of $\det(\partial\Phi)$ with negative real part as $\lambda_1 = -1$, $\lambda_2 = -0.936$, $\lambda_3 = -0.456 - 0.927i$, $\lambda_4 = -0.456 + 0.927i$.

The basis matrix V_i for Ker $(\partial\Phi(\lambda_i))$, $i = 1, \ldots, 4$ corresponds to the i-th column of the matrix

$$\begin{pmatrix} 0.707 & -0.738 & -0.852 & -0.852 \\ 0.707 & -0.674 & 0.133 - 0.505\, i & 0.133 + 0.505\, i \end{pmatrix}$$

and the eigenvalues of the associated Pick matrix are

$$-1.865 \quad -1.601 \quad -0.198 \quad -0.001$$

Since $T_\Phi < 0$, there exists a strictly γ-contracting controller with $\gamma = 0.7$, and therefore there exists a $J_{1,1}$-factorization of $\partial\Phi$.

Observe that a minimal state map for M is

$$X = \begin{pmatrix} 0 & 1 \\ 0 & \xi \\ 0 & \xi^2 \\ 1 & 0 \end{pmatrix}$$

The matrix $V = \begin{pmatrix} X(\lambda_1)V_1 & \dots & X(\lambda_4)V_4 \end{pmatrix}$ is

$$V = \begin{pmatrix} 0.707 & -0.674 & 0.133 - 0.505\ i & 0.133 + 0.505\ i \\ -0.707 & 0.631 & -0.529 + 0.107\ i & -0.529 - 0.107\ i \\ 0.707 & -0.590 & 0.341 + 0.442\ i & 0.341 - 0.442\ i \\ 0.707 & -0.738 & -0.852 & -0.852 \end{pmatrix}$$

We denote with $\tilde{C}$ the coefficient matrix of

$$\Phi(\zeta, \eta) - (\zeta + \eta) X^T(\zeta)(V^*)^{-1} T_\Phi V^{-1} X(\eta)$$

The nonzero eigenvalues of $\tilde{C}$ are 26.840 and -3.571, one positive and the other one negative, as expected. A factorization $\tilde{F}^T J_{1,1} \tilde{F}$ of $\tilde{C}$ can be obtained by means of a singular value decomposition of $\tilde{C}$, with

$$\tilde{F} = \begin{pmatrix} -0.550 & 1.696 & -0.406 & 3.549 & 0 & 3.108 & 0 & 1.112 \\ 0.0491 & 1.370 & 0.930 & 0.248 & 0 & -0.727 & 0 & -0.486 \end{pmatrix}$$

The J-spectral factor

$$\begin{pmatrix} -0.550 - 0.406\ \xi & 1.696 + 3.549\ \xi + 3.108\ \xi^2 + 1.112\ \xi^3 \\ 0.0491 + 0.93\ \xi & 1.37 + 0.248\ \xi - 0.727\ \xi^2 - 0.486\ \xi^3 \end{pmatrix}$$

is Hurwitz (the roots of its determinant coincide with the left-half plane roots of $\det(\partial\Phi)$). Moreover,

$$\begin{pmatrix} D \\ F_+ \end{pmatrix} = \begin{pmatrix} 1 + \xi & 0 \\ -0.550 - 0.406\ \xi & 1.696 + 3.549\ \xi + 3.108\ \xi^2 + 1.112\ \xi^3 \end{pmatrix}$$

is also Hurwitz, since the roots of its determinant are -1.19, -1, $-0.803 - 0.798i$, $-0.803 + 0.798i$.

The desired controller is therefore described by

$$-0.550\ell_1 - 0.406\ \frac{\mathrm{d}\ell_1}{\mathrm{d}t} + 1.696\ell_2 + 3.549\ \frac{\mathrm{d}\ell_2}{\mathrm{d}t} + 3.108\frac{\mathrm{d}^2\ell_2}{\mathrm{d}t^2} + 1.112\ \frac{\mathrm{d}^3\ell_2}{\mathrm{d}t^3} = 0$$

$\square$